

Counting (k, l) -sumsets in groups of prime order

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Abstract. A subset A of a group \mathbf{G} is called (k, l) -*sumset*, if $A = kB - lB$ for some $B \subseteq \mathbf{G}$, where $kB - lB = \{x_1 + \dots + x_k - x_{k+1} - \dots - x_{k+l} : x_1, \dots, x_{k+l} \in B\}$. Upper and lower bounds for the number (k, l) -sumsets in groups of prime order are provided.

1 Introduction

Let p be a prime number and k, l be nonnegative integers with $k+l \geq 2$. Write \mathbf{Z}_p for the group of residues modulo p . A subset $A \subseteq \mathbf{Z}_p$ is called (k, l) -*sumset*, if $A = kB - lB$ for some $B \subseteq \mathbf{Z}_p$, where $kB - lB = \{x_1 + \dots + x_k - x_{k+1} - \dots - x_{k+l} : x_1, \dots, x_{k+l} \in B\}$. Write $\mathbf{SS}_{k,l}(\mathbf{Z}_p)$ for the collection of (k, l) -sumsets in \mathbf{Z}_p .

B. Green and I. Ruzsa in [1] proved

$$p^2 2^{p/3} \ll |\mathbf{SS}_{2,0}(\mathbf{Z}_p)| \leq 2^{p/3 + \theta(p)}$$

where $\theta(p)/p \rightarrow 0$ as $p \rightarrow \infty$ and $\theta(p) \ll p(\log \log p)^{2/3}(\log p)^{-1/9}$ (hereafter logarithms are to base two).

The aim of this work is to obtain bounds for the number $|\mathbf{SS}_{k,l}(\mathbf{Z}_p)|$. We prove

Theorem 1 *Let p be a prime number and k, l be nonnegative integers with $k+l \geq 2$. Then there exists a positive constant $C_{k,l}$ such that*

$$C_{k,l} 2^{p/(2(k+l)-1)} \leq |\mathbf{SS}_{k,l}(\mathbf{Z}_p)| \leq 2^{(p/(k+l+1)) + (k+l-2) + o(p)}. \quad (1)$$

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2 Definitions and auxiliary results

Let \mathbf{R} be the set of real numbers, $f_i : \mathbf{Z}_p \rightarrow \mathbf{R}$, $i = 1, \dots, m$, and $x \in \mathbf{Z}_p$. We set

$$\begin{aligned} & (f_1 * \dots * f_m)(x) = \\ &= \sum_{x_1 \in \mathbf{Z}_p} \dots \sum_{x_{m-1} \in \mathbf{Z}_p} f_1(x_1) \dots f_{m-1}(x_{m-1}) f_m(x - x_1 - \dots - x_{m-1}) \end{aligned} \quad (2)$$

and

$$\widehat{f}(x) = \sum_{y \in \mathbf{Z}_p} f(y) e^{2\pi i \frac{xy}{p}}.$$

The function $\widehat{f}(x)$ is called *Fourier transform* of f .

Lemma 2 *We have*

$$(f_1 * \dots * f_m)(x) = \widehat{f_1}(x) \dots \widehat{f_m}(x). \quad (3)$$

Proof. By definition

$$\begin{aligned} & (\widehat{f_1 * \dots * f_m})(x) = \sum_{y \in \mathbf{Z}_p} (f_1 * \dots * f_m)(y) e^{2\pi i \frac{yx}{p}} = \\ &= \sum_{y \in \mathbf{Z}_p} \sum_{y_1 \in \mathbf{Z}_p} \dots \sum_{y_{m-1} \in \mathbf{Z}_p} f_1(y_1) \dots f_{m-1}(y_{m-1}) \times \\ & \times f_m(y - y_1 - \dots - y_{m-1}) \cdot e^{2\pi i \frac{yx}{p}} \dots e^{2\pi i \frac{y_{m-1}x}{p}} \cdot e^{2\pi i \frac{(y - y_1 - \dots - y_{m-1})x}{p}} = \\ &= \sum_{y_1 \in \mathbf{Z}_p} f_1(y_1) \cdot e^{2\pi i \frac{y_1 x}{p}} \dots \sum_{y_{m-1} \in \mathbf{Z}_p} f_{m-1}(y_{m-1}) \cdot e^{2\pi i \frac{y_{m-1} x}{p}} \times \\ & \times \sum_{y \in \mathbf{Z}_p} f_m(y - y_1 - \dots - y_{m-1}) \cdot e^{2\pi i \frac{(y - y_1 - \dots - y_{m-1})x}{p}} = \widehat{f_1}(x) \dots \widehat{f_m}(x). \end{aligned}$$

□

Denote the characteristic function of a set A by $\chi_A(x)$. Let A_1, \dots, A_m be non-empty subsets of \mathbf{Z}_p . Then $(\chi_{A_1} * \dots * \chi_{A_m})(x)$ will be the number of vectors $(x_1, \dots, x_m) \in A_1 \times \dots \times A_m$ such that $x \equiv x_1 + \dots + x_m \pmod{p}$. Set $A_1 + \dots + A_m = \{x_1 + \dots + x_m \pmod{p} : x_1 \in A_1, \dots, x_m \in A_m\}$. We define $S_{h,m}(A_1, \dots, A_m) = \{x \in \mathbf{Z}_p : (\chi_{A_1} * \dots * \chi_{A_m})(x) \geq h\}$, where $h > 0$. Further, for any integer i and any $A \subseteq \mathbf{Z}_p$ denote the set $\underbrace{A + \dots + A}_i$ by iA , and the set $\{p - x : x \in A\}$ by $-A$.

Theorem 3 (Cauchy-Davenport, [2]). *Let A_1, \dots, A_m be non-empty subsets of \mathbb{Z}_p . Then $|A_1 + \dots + A_m| \geq \min(p, |A_1| + \dots + |A_m| - (m - 1))$.*

Theorem 4 (Pollard, [3]). *Let A_1, A_2 be non-empty subsets of \mathbb{Z}_p . Then*

$$|S_{1,2}(A_1, A_2)| + \dots + |S_{t,2}(A_1, A_2)| \geq t \min(p, |A_1| + |A_2| - t),$$

where $t \leq \min(|A_1|, |A_2|)$.

Theorems 3, 4 imply the following two statements.

Lemma 5 *Let A_1, \dots, A_m non-empty subsets of \mathbb{Z}_p . Then*

$$\begin{aligned} |S_{1,m}(A_1, \dots, A_m)| + \dots + |S_{t,m}(A_1, \dots, A_m)| &\geq \\ &\geq t \min(p, |A_1| + \dots + |A_m| - t - m + 2), \end{aligned}$$

where $t \leq \min(|A_1|, \dots, |A_m|)$.

Proof. Without loss of generality we assume $|A_1| = \min(|A_1|, \dots, |A_m|)$. By Theorem 4 we have

$$\begin{aligned} |S_{1,2}(A_1, (A_2 + \dots + A_m))| + \dots + |S_{t,2}(A_1, (A_2 + \dots + A_m))| &\geq \\ &\geq t \min(p, |A_1| + |A_2 + \dots + A_m| - t), \end{aligned} \quad (4)$$

where $t \leq |A_1|$.

On the other hand by Theorem 3 we have

$$|A_2 + \dots + A_m| \geq \min(p, |A_2| + \dots + |A_m| - (m - 2)). \quad (5)$$

Substituting (5) in (4), we obtain

$$\begin{aligned} |S_{1,m}(A_1, \dots, A_m)| + \dots + |S_{t,m}(A_1, \dots, A_m)| &\geq \\ &\geq |S_{1,2}(A_1, (A_2 + \dots + A_m))| + \dots + |S_{t,2}(A_1, (A_2 + \dots + A_m))| \geq \\ &\geq t \min(p, |A_1| + \dots + |A_m| - t - m + 2). \end{aligned}$$

□

Lemma 6 *Let A_1, \dots, A_m be non-empty subsets of \mathbb{Z}_p and $h \leq \min(|A_1|, \dots, |A_m|)$. Then*

$$|S_{h,m}(A_1, \dots, A_m)| \geq \min(p, |A_1| + \dots + |A_m| - m + 2) - 2(hp)^{1/2}.$$

Proof. Note that $|S_{i,m}(A_1, \dots, A_m)| \geq |S_{j,m}(A_1, \dots, A_m)|$ for $i \leq j$. Choose $h \leq t \leq \min(|A_1|, \dots, |A_m|)$. By Lemma 5 we have

$$\begin{aligned} & t \min(p, |A_1| + \dots + |A_m| - t - m + 2) \leq \\ & \leq |S_{1,m}(A_1, \dots, A_m)| + \dots + |S_{t,m}(A_1, \dots, A_m)| \leq \\ & \leq hp + t|S_{h,m}(A_1, \dots, A_m)|. \end{aligned}$$

Putting $t = (hp)^{1/2}$, we get

$$\begin{aligned} & \min(p, |A_1| + \dots + |A_m| - m + 2) - 2(hp)^{1/2} \leq \\ & \leq \min(p, |A_1| + \dots + |A_m| - m - (hp)^{1/2} + 2) - (hp)^{1/2} \leq \\ & \leq |S_{h,m}(A_1, \dots, A_m)|. \end{aligned}$$

□

Lemma 7 Set $T_{r,s}(\mathbf{Z}_p) = \{A \subset \mathbf{Z}_p : |A| \leq p/(r+1)s\}$. Then there exists s such that

$$|T_{r,s}(\mathbf{Z}_p)| \leq 2^{p/(r+1)}. \quad (6)$$

Proof. Let n, m be positive integers, $1 \leq m \leq n$. Then (see Lemma 6.8, [4])

$$\sum_{0 \leq i \leq m} \binom{n}{i} \leq \left(\frac{en}{m}\right)^m. \quad (7)$$

We choose s such that

$$es(r+1) \leq 2^s. \quad (8)$$

Then by (7) we have (putting $n = p$ and $m = p/(r+1)s$)

$$|T_{r,s}(\mathbf{Z}_p)| = \sum_{0 \leq i \leq p/(r+1)s} \binom{p}{i} \leq (es(r+1))^{p/(r+1)s} \leq (2^s)^{p/(r+1)s} = 2^{p/(r+1)}.$$

□

Let L be a positive integer. For each $y \in \{0, \dots, p-1\}$ we define a partition $\mathbf{R}_{y,L}$ of \mathbf{Z}_p on the intervals of the form $J_i^y = \{(iL+1+y) \pmod{p}, \dots, ((i+1)L+y) \pmod{p}\}$, $0 \leq i \leq \lfloor p/L \rfloor - 1$. All intervals are J_i^y of $\mathbf{R}_{y,L}$ have length L , and the set $J_y = \mathbf{Z}_p \setminus \bigcup_i J_i^y$ has cardinality $p - L\lfloor p/L \rfloor < L$. The set J_y is called *remainder* partition $\mathbf{R}_{y,L}$. In what follows we fix $y \in \{0, \dots, p-1\}$ and consider the corresponding partition $\mathbf{R}_{y,L}$. For every $A \subseteq \mathbf{Z}_p$ and any integer d define $d \star A = \{da \pmod{p} : a \in A\}$. The set $d \star A$ is called *dilation* of A . The set $A \subseteq \mathbf{Z}_p$ is called *L-granular* (see [1]), if some dilation of A is a union of some of the intervals J_i^y (other than remainder). We denote the family of L -granular subsets of \mathbf{Z}_p by $\mathbf{G}_L(\mathbf{Z}_p)$.

Lemma 8 *We have*

$$|G_L(\mathbf{Z}_p)| \leq p2^{p/L}. \quad (9)$$

Proof. Denote the number of subsets of intervals (other than remainder) of the partition $\mathbf{R}_{y,L}$ of \mathbf{Z}_p by $g(\mathbf{R}_{y,L})$, and the number of different partitions $\mathbf{R}_{y,L}$ of \mathbf{Z}_p by $r(L)$. It is obvious that

$$|G_L(\mathbf{Z}_p)| \leq g(\mathbf{R}_{y,L})r(L). \quad (10)$$

Note that the number of intervals (other than remainder) of the partition $\mathbf{R}_{y,L}$ of \mathbf{Z}_p is equal to $\lfloor p/L \rfloor$, and the number of different partitions $\mathbf{R}_{y,L}$ of \mathbf{Z}_p is at most p . This and (10) imply the inequality (9). \square

Lemma 9 *Let $A \subseteq \mathbf{Z}_p$ have size αp , and let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be positive real numbers and $L > 0$, k, l be nonnegative integers satisfying $k + l \geq 2$. Suppose that*

$$p > (\sqrt{8(k+l)L})^{4^{2(k+l)}\alpha^{2(k+l-1)}\varepsilon_1^{-2(k+l)}\varepsilon_2^{-2(k+l-1)}\varepsilon_3^{-1}}. \quad (11)$$

Then there exists a set $A' \subseteq \mathbf{Z}_p$ with the following properties:

- (i) A' is L -granular;
- (ii) $|A \setminus A'| \leq \varepsilon_1 p$;
- (iii) *the set $kA - lA$ contains all $x \in \mathbf{Z}_p$ for which*

$$\underbrace{(\chi_{A'} * \cdots * \chi_{A'})}_k * \underbrace{(\chi_{-A'} * \cdots * \chi_{-A'})}_l(x) \geq (\varepsilon_2 p)^{k+l-1},$$
with at most $\varepsilon_3 p$ exceptions.

Proof. Let $h \in \{0, \dots, p-1\}$, and $\mathbf{R}_{h,L}$ be partition of \mathbf{Z}_p .

(i) For given set $A \subset \mathbf{Z}_p$ we define $A' \subset \mathbf{Z}_p$ as the union of intervals J_i^h of the partition $\mathbf{R}_{h,L}$, such that $|A \cap J_i^h| \geq \varepsilon_1 L/2$. From the definition it follows that A' is L -granular. It is easy to see that $(-A)' = -(A')$.

(ii) Let $x \in A \setminus A'$. Then either $x \in J_h$ or $x \in A \cap J_i^h$, ($i = 0, \dots, \lfloor p/L \rfloor - 1$), and $|A \cap J_i^h| \leq \varepsilon_1 L/2$. In the first case we have $|J_h| < L$, and inequality (11) implies $L \leq \varepsilon_1 p/2$. Thus,

$$|A \setminus A'| \leq \frac{\varepsilon_1 L}{2} \cdot \frac{p}{L} + L \leq \varepsilon_1 p.$$

(iii) Let $\widehat{\chi_A}(x)$ be the Fourier transform of the characteristic function χ_A of A , so that

$$\widehat{\chi_A}(x) = \sum_{y \in \mathbf{Z}_p} \chi_A(y) e^{2\pi i \frac{yx}{p}} = \sum_{y \in A} e^{2\pi i \frac{yx}{p}}$$

for all $x \in \mathbf{Z}_p$. Take $\delta = 4^{-(k+l)} \varepsilon_1^{k+l} \varepsilon_2^{k+l-1} \varepsilon_3^{1/2} \alpha^{-(k+l)+3/2}$, where $\varepsilon_1, \varepsilon_2$ and ε_3 are from inequality (11). Set $\mathbf{D} = \{x \neq 0 : |\widehat{\chi}_\Lambda(x)| \geq \delta p\}$. We define the function $f(x)$ as follows:

$$f(x) = \frac{1}{2L-1} \sum_{j=-(L-1)}^{L-1} e^{2\pi i \frac{jqx}{p}}.$$

In the future we will show that there exists $q \in \mathbf{Z}_p \setminus \{0\}$ such that for all $x \in \mathbf{Z}_p$ it holds

$$|\widehat{\chi}_\Lambda(x)| |1 - f^{k+l}(x)| \leq \delta p. \quad (12)$$

The inequality (12) obviously holds for the case $x = 0$, since $f(0) = 1$, as well as for the case $|\widehat{\chi}_\Lambda(x)| \leq \delta p$, since $f(x) \in [-1, 1]$. Thus, it remains to show the existence of q such that the inequality (12) holds for all $x \in \mathbf{D}$. First we estimate the value of $1 - f(x)$. Denote by $\langle x \rangle$ the distance from x to the nearest integer. We use the fact that $1 - \cos(2\pi x) \leq 2\pi^2 \langle x \rangle^2$. Then

$$\begin{aligned} 1 - f(x) &= \frac{2}{2L-1} \sum_{j=1}^{L-1} \left(1 - \cos \frac{2\pi j qx}{p} \right) \leq \frac{4\pi^2}{2L-1} \sum_{j=1}^{L-1} \left\langle \frac{j qx}{p} \right\rangle^2 \leq \\ &\leq \frac{4\pi^2}{2L-1} \left\langle \frac{qx}{p} \right\rangle^2 \sum_{j=1}^{L-1} j^2 \leq \frac{2\pi^2 L^2}{3} \left\langle \frac{qx}{p} \right\rangle^2. \end{aligned} \quad (13)$$

Recall that for $|x| \leq 1$

$$1 - x^m = (1 - x)(1 + x + x^2 + \dots + x^{m-1}) \leq m(1 - x). \quad (14)$$

From (13) and (14) it follows

$$|\widehat{\chi}_\Lambda(x)| |1 - f^{k+l}(x)| \leq (k+l) |\widehat{\chi}_\Lambda(x)| |1 - f(x)| \leq 8(k+l) L^2 \langle qx/p \rangle^2 |\widehat{\chi}_\Lambda(x)|.$$

Note that if the inequality

$$\left\langle \frac{qx}{p} \right\rangle \leq \frac{1}{\sqrt{8(k+l)L}} \left(\frac{\delta p}{|\widehat{\chi}_\Lambda(x)|} \right)^{1/2} \quad (15)$$

holds for some $q \in \mathbf{Z}_p \setminus \{0\}$ and for all $x \in \mathbf{D}$ then the inequality (12) also holds. Now we will prove that such q exists. By definition, we have

$$\langle qx/p \rangle = \min\{(qx \bmod p)/p, (p - qx \bmod p)/p\}.$$

Set $|\mathbf{D}| = d$, $\mathbf{D} = \{r_1, \dots, r_d\}$. We denote $\mathbf{a}_i = (1/\sqrt{8(k+l)L})(\delta p / |\widehat{\chi_\lambda}(r_i)|)^{1/2}$. Then the inequality (15) can be rewritten as

$$\min\{qr_i \pmod{p}, p - qr_i \pmod{p}\} \leq p\mathbf{a}_i, \text{ where } i = 1, \dots, d. \quad (16)$$

Denote the set $\{(x_1, \dots, x_d) : x_1, \dots, x_d \in \mathbf{Z}_p\}$ by \mathbf{Z}_p^d . We split \mathbf{Z}_p^d on disjoint subsets

$$\mathbf{Z}_p^d = \bigcup_{(i_1, \dots, i_d)} \mathbf{Q}_{i_1, \dots, i_d},$$

where

$$\mathbf{Q}_{i_1, \dots, i_d} = \{(x_1, \dots, x_d) : i_j p \mathbf{a}_j < x_j \leq (i_j + 1)p \mathbf{a}_j, j = 1, \dots, d\}.$$

Let μ_d be number of different sets of $\mathbf{Q}_{i_1, \dots, i_d}$. Using the fact that $0 \leq i_j \leq 1/\mathbf{a}_j - 1$, $j = 1, \dots, d$, we have

$$\mu_d \leq \prod_{i=1}^d \frac{1}{\mathbf{a}_i}.$$

Let us consider the following $p - 1$ elements of \mathbf{Z}_p^d :

$$(qr_1 \pmod{p}, \dots, qr_d \pmod{p}), \text{ where } r_1, \dots, r_d \in \mathbf{D}, \quad q = 1, \dots, p - 1.$$

We show that if

$$p > \prod_{i=1}^d \frac{1}{\mathbf{a}_i}, \quad (17)$$

then there exists q such that for all $r_i \in \mathbf{D}$, $i = 1, \dots, d$, the inequality (16) holds. We consider two cases:

(A) If $\mu_d = p - 1$, then we take $q = q_0$, where $q_0 \in \mathbf{Z}_p \setminus \{0\}$ such that

$$(q_0 r_1 \pmod{p}, \dots, q_0 r_d \pmod{p}) \in \mathbf{Q}_{0, \dots, 0}.$$

(B) If $\mu_d < p - 1$, then by pigeonhole principle, there are $q_1, q_2 \in \mathbf{Z}_p \setminus \{0\}$ such that the vectors $(q_1 r_1 \pmod{p}, \dots, q_1 r_d \pmod{p})$ and $(q_2 r_1 \pmod{p}, \dots, q_2 r_d \pmod{p})$ belong to the same set of $\mathbf{Q}_{i_1, \dots, i_d}$. Obviously, when $q = q_1 - q_2$ the inequality (16) holds.

We now show that inequality (17) is a consequence of (11). Indeed, by the Parseval's identity, we have

$$p^{-1} \left(\sum_{x \in \mathbf{D}} |\widehat{\chi_\lambda}(x)|^2 + \sum_{x \in \mathbf{Z}_p \setminus \mathbf{D}} |\widehat{\chi_\lambda}(x)|^2 \right) = \sum_{x \in \mathbf{Z}_p} |\chi_\lambda(x)|^2 = \alpha p. \quad (18)$$

From (18) it follows

$$\sum_{x \in \mathbf{D}} |\widehat{\chi_A}(x)|^2 \leq \alpha p^2. \quad (19)$$

From (19) and the arithmetic and geometric mean inequality, we get

$$\left(\prod_{x \in \mathbf{D}} |\widehat{\chi_A}(x)|^2 \right)^{1/d} \leq \frac{1}{d} \sum_{x \in \mathbf{D}} |\widehat{\chi_A}(x)|^2 \leq \frac{\alpha p^2}{d}.$$

i.e.

$$\prod_{x \in \mathbf{D}} |\widehat{\chi_A}(x)| \leq \left(\frac{\alpha p^2}{d} \right)^{d/2}. \quad (20)$$

From (20) we get

$$(\sqrt{8(k+l)L})^d \left(\prod_{x \in \mathbf{D}} \frac{|\widehat{\chi_A}(x)|}{\delta p} \right)^{1/2} \leq (\sqrt{8(k+l)L} \alpha^{1/4} \delta^{-1/2} d^{-1/4})^d. \quad (21)$$

It is easy to see that the right-hand side of (21) is an increasing function of d in the range $d < 64(k+l)^2 L^4 \alpha / \delta^2 e$.

On the other hand, from (19) we have $d \delta^2 p^2 \leq \alpha p^2$. Hence, $d \leq \alpha / \delta^2$. Consequently

$$(\sqrt{8(k+l)L} \alpha^{1/4} \delta^{-1/2} d^{-1/4})^d \leq (\sqrt{8(k+l)L})^{\alpha / \delta^2}.$$

Recall that $\delta = 4^{-(k+l)} \varepsilon_1^{k+l} \varepsilon_2^{k+l-1} \varepsilon_3^{1/2} \alpha^{-(k+l)+3/2}$. From this it follows that there exists q such that the inequality (12) holds. Moreover, without loss of generality we can assume $q = 1$ (this can be achieved by selecting an appropriate dilation of the set A).

Define two functions $\chi_1(x)$ and $\chi_2(x)$ as follows:

$$\chi_1(x) = \frac{1}{|\mathcal{J}|} (\chi_A * \chi_{\mathcal{J}})(x),$$

$$\chi_2(x) = \frac{1}{|\mathcal{J}|} (\chi_{-A} * \chi_{\mathcal{J}})(x),$$

where $\mathcal{J} = \{-(L-1), \dots, L-1\}$. From (2) it follows that

$$\chi_1(x) = \frac{1}{|\mathcal{J}|} |A \cap (\mathcal{J} + x)|, \quad (22)$$

$$\chi_2(x) = \frac{1}{|\mathcal{J}|} |(-A) \cap (\mathcal{J} + x)|, \quad (23)$$

and from (3) we have $\widehat{\chi}_1(x) = \widehat{\chi_A}(x)f(x)$ and $\widehat{\chi}_2(x) = \widehat{\chi_{-A}}(x)f(x)$. Hence, by Parseval's identity and from (3) we get

$$\begin{aligned} & \sum_{x \in \mathbf{Z}_p} \left| \underbrace{(\chi_A * \cdots * \chi_A)}_k * \underbrace{(\chi_{-A} * \cdots * \chi_{-A})}_l(x) - \underbrace{(\chi_1 * \cdots * \chi_1)}_k * \underbrace{(\chi_2 * \cdots * \chi_2)}_l(x) \right|^2 = \\ &= p^{-1} \sum_{x \in \mathbf{Z}_p} \left| \underbrace{(\widehat{\chi_A} * \cdots * \widehat{\chi_A})}_k * \underbrace{(\widehat{\chi_{-A}} * \cdots * \widehat{\chi_{-A}})}_l(x) - \underbrace{(\widehat{\chi_1} * \cdots * \widehat{\chi_1})}_k * \underbrace{(\widehat{\chi_2} * \cdots * \widehat{\chi_2})}_l(x) \right|^2 = \\ &= p^{-1} \sum_{x \in \mathbf{Z}_p} \left| \widehat{\chi_A}^k(x) \widehat{\chi_{-A}}^l(x) - \widehat{\chi_1}^k(x) \widehat{\chi_2}^l(x) \right|^2 = \\ &= p^{-1} \sum_{x \in \mathbf{Z}_p} |\widehat{\chi_A}(x)|^{2k} |\widehat{\chi_{-A}}(x)|^{2l} |1 - f^{k+l}(x)|^2 \leq \\ &\leq p^{-1} \left(\sup_{x \in \mathbf{Z}_p} |\widehat{\chi_A}(x)|^{k-1} |\widehat{\chi_{-A}}(x)|^l |1 - f^{k+l}(x)| \right)^2 \sum_{x \in \mathbf{Z}_p} |\widehat{\chi_A}(x)|^2. \quad (24) \end{aligned}$$

We have

$$|\widehat{\chi_A}(x)| = \left| \sum_{y \in \mathbf{Z}_p} \chi_A(y) e^{2\pi i \frac{yx}{p}} \right| = \left| \sum_{y \in A} e^{2\pi i \frac{yx}{p}} \right| \leq \sum_{y \in A} \left| e^{2\pi i \frac{yx}{p}} \right| = \alpha p, \quad (25)$$

$$|\widehat{\chi_{-A}}(x)| = \left| \sum_{y \in \mathbf{Z}_p} \chi_{-A}(y) e^{2\pi i \frac{yx}{p}} \right| = \left| \sum_{y \in -A} e^{2\pi i \frac{yx}{p}} \right| \leq \sum_{y \in -A} \left| e^{2\pi i \frac{yx}{p}} \right| = \alpha p. \quad (26)$$

From (12), (18), (24), (25) and (26) it follows

$$\begin{aligned} & \sum_{x \in \mathbf{Z}_p} \left| \underbrace{(\chi_A * \cdots * \chi_A)}_k * \underbrace{(\chi_{-A} * \cdots * \chi_{-A})}_l(x) - \underbrace{(\chi_1 * \cdots * \chi_1)}_k * \underbrace{(\chi_2 * \cdots * \chi_2)}_l(x) \right|^2 \leq \\ &\leq \left(\sup_{x \in \mathbf{Z}_p} |\widehat{\chi_A}(x)| |1 - f^{k+l}(x)| \right)^2 \alpha^{2(k+l)-3} p^{2(k+l)-3} \leq \end{aligned}$$

$$\leq \alpha^{2(k+l)-3} \delta^2 p^{2(k+l)-1}. \quad (27)$$

Suppose that $x \in A'$ ($x \in -A'$). Then there exists an interval \mathcal{I} of length L such that $\mathcal{I} \subseteq \{x - (L-1), \dots, x + (L-1)\}$ and $x \in \mathcal{I}$. From definition of A' ($-A'$) it follows that $|\mathcal{I} \cap A| \geq \varepsilon_1 L/2$ ($|\mathcal{I} \cap (-A)| \geq \varepsilon_1 L/2$). From the definition of $\chi_1(x)$ ($\chi_2(x)$) it follows that $\chi_1(x) \geq \varepsilon_1/4$ ($\chi_2(x) \geq \varepsilon_1/4$). Observe, that $\chi_1(x) \geq \varepsilon_1 \chi_{A'}(x)/4$ and $\chi_2(x) \geq \varepsilon_1 \chi_{-A'}(x)/4$ hold for all $x \in \mathbf{Z}_p$. From this and (2) it follows that

$$\begin{aligned} & (\underbrace{\chi_1 * \dots * \chi_1}_k * \underbrace{\chi_2 * \dots * \chi_2}_l)(x) \geq \\ & \geq \varepsilon_1^{k+l} (\underbrace{\chi_{A'} * \dots * \chi_{A'}}_k * \underbrace{\chi_{-A'} * \dots * \chi_{-A'}}_l)(x) / 4^{k+l} \end{aligned} \quad (28)$$

for all $x \in \mathbf{Z}_p$. In the case

$$(\underbrace{\chi_{A'} * \dots * \chi_{A'}}_k * \underbrace{\chi_{-A'} * \dots * \chi_{-A'}}_l)(x) \geq (\varepsilon_2 p)^{k+l-1}, \quad (29)$$

by (28) we have

$$(\underbrace{\chi_1 * \dots * \chi_1}_k * \underbrace{\chi_2 * \dots * \chi_2}_l)(x) \geq \varepsilon_1^{k+l} (\varepsilon_2 p)^{k+l-1} / 4^{k+l}. \quad (30)$$

Now we show that the number of elements $x \in \mathbf{Z}_p$ such that satisfying (29) and $(\underbrace{\chi_A * \dots * \chi_A}_k * \underbrace{\chi_{-A} * \dots * \chi_{-A}}_l)(x) = 0$, does not exceed $\varepsilon_3 p$. Denote the set of such elements by \mathbf{F} . Observe, that for every $x \in \mathbf{F}$

$$\begin{aligned} & |(\underbrace{\chi_A * \dots * \chi_A}_k * \underbrace{\chi_{-A} * \dots * \chi_{-A}}_l)(x) - (\underbrace{\chi_1 * \dots * \chi_1}_k * \underbrace{\chi_2 * \dots * \chi_2}_l)(x)|^2 \geq \\ & \geq \frac{\varepsilon_1^{2(k+l)} \varepsilon_2^{2(k+l-1)} p^{2(k+l-1)}}{4^{2(k+l)}}. \end{aligned} \quad (31)$$

By (27) and (31)

$$\begin{aligned} & \alpha^{2(k+l)-3} \delta^2 p^{2(k+l)-1} \geq \\ & \geq \sum_{x \in \mathbf{Z}_p} \left| (\underbrace{\chi_A * \dots * \chi_A}_k * \underbrace{\chi_{-A} * \dots * \chi_{-A}}_l)(x) - (\underbrace{\chi_1 * \dots * \chi_1}_k * \underbrace{\chi_2 * \dots * \chi_2}_l)(x) \right|^2 = \end{aligned}$$

$$\begin{aligned}
&= \sum_{x \in \mathbf{F}} \left| \left(\underbrace{\chi_A * \cdots * \chi_A}_k * \underbrace{\chi_{-A} * \cdots * \chi_{-A}}_l \right)(x) - \left(\underbrace{\chi_1 * \cdots * \chi_1}_k * \underbrace{\chi_2 * \cdots * \chi_2}_l \right)(x) \right|^2 + \\
&+ \sum_{x \in (\mathbf{Z}_p \setminus \mathbf{F})} \left| \left(\underbrace{\chi_A * \cdots * \chi_A}_k * \underbrace{\chi_{-A} * \cdots * \chi_{-A}}_l \right)(x) - \left(\underbrace{\chi_1 * \cdots * \chi_1}_k * \underbrace{\chi_2 * \cdots * \chi_2}_l \right)(x) \right|^2 \\
&\geq |\mathbf{F}| \frac{\varepsilon_1^{2(k+l)} \varepsilon_2^{2(k+l-1)} p^{2(k+l-1)}}{4^{2(k+l)}} + \\
&+ \sum_{x \in (\mathbf{Z}_p \setminus \mathbf{F})} \left| \left(\underbrace{\chi_A * \cdots * \chi_A}_k * \underbrace{\chi_{-A} * \cdots * \chi_{-A}}_l \right)(x) - \left(\underbrace{\chi_1 * \cdots * \chi_1}_k * \underbrace{\chi_2 * \cdots * \chi_2}_l \right)(x) \right|^2.
\end{aligned}$$

This implies

$$|\mathbf{F}| \leq \frac{4^{2(k+l)} \alpha^{2(k+l)-3} \delta^2}{\varepsilon_1^{2(k+l)} \varepsilon_2^{2(k+l-1)}} p \leq \varepsilon_3 p.$$

□

3 The proof of Theorem 1

3.1 The upper bound

Let k, l be nonnegative integers with $k + l \geq 2$. Suppose that s satisfies $es(k + l + 1) \leq 2^s$. We divide a partition of $\mathbf{SS}_{k,l}(\mathbf{Z}_p)$ into two parts:

$$\mathbf{SS}_{k,l}(\mathbf{Z}_p) = \mathbf{SS}'_{k,l,s}(\mathbf{Z}_p) \cup \mathbf{SS}''_{k,l,s}(\mathbf{Z}_p), \quad (32)$$

where

$$\mathbf{SS}'_{k,l,s}(\mathbf{Z}_p) = \{B \in \mathbf{SS}_{k,l}(\mathbf{Z}_p) : B = kA - lA \text{ and } |A| \leq p/(k + l + 1)s\},$$

$$\mathbf{SS}''_{k,l,s}(\mathbf{Z}_p) = \{B \in \mathbf{SS}_{k,l}(\mathbf{Z}_p) : B = kA - lA \text{ and } |A| > p/(k + l + 1)s\}.$$

It is obvious that

$$|\mathbf{SS}_{k,l}(\mathbf{Z}_p)| \leq |\mathbf{SS}'_{k,l,s}(\mathbf{Z}_p)| + |\mathbf{SS}''_{k,l,s}(\mathbf{Z}_p)|. \quad (33)$$

Since every set $A \subseteq \mathbf{Z}_p$ generates one set of the form $kA - lA$ we obtain

$$|\mathbf{SS}'_{k,l,s}(\mathbf{Z}_p)| \leq |\mathbf{T}_{k+l,s}(\mathbf{Z}_p)|. \quad (34)$$

By (7) and (34) we have

$$|\mathbf{SS}'_{k,l,s}(\mathbf{Z}_p)| \leq 2^{p/(k+l+1)}. \quad (35)$$

Now we prove an upper bound for $|\mathbf{SS}''_{k,l,s}(\mathbf{Z}_p)|$. Suppose that the cardinality of $A \subseteq \mathbf{Z}_p$ is larger than $p/(k+l+1)s$. Let p be a prime number such that for some nonnegative integers $k, l, L > 0$ and positive real numbers $\varepsilon_1, \varepsilon_2$ and ε_3 the condition (11) is fulfilled. By Lemma 9 there exists a subset A' with properties (i) – (iii). We estimate the number of (k, l) -sumsets $kA - lA$ by counting pairs $(A', kA - lA)$.

Now let $A' \in \mathbf{G}_L(\mathbf{Z}_p)$ be given. For any subset $C \subseteq \mathbf{Z}_p$ we denote by \overline{C} the complement of the subset C in \mathbf{Z}_p .

If $|A'| \geq p/(k+l+1)$, then from (iii) of Lemma 9 we obtain that $\overline{kA - lA}$ is a subset of the union of the set $\overline{S_{(\varepsilon_2 p)^{k+l-1}, k+l}(\underbrace{\chi_{A'}, \dots, \chi_{A'}}_k, \underbrace{\chi_{-A'}, \dots, \chi_{-A'}}_l)}$

and a set of cardinality not exceeding $\varepsilon_3 p$. By Lemma 6 we have

$$\begin{aligned} |S_{(\varepsilon_2 p)^{k+l-1}, k+l}(\underbrace{\chi_{A'}, \dots, \chi_{A'}}_k, \underbrace{\chi_{-A'}, \dots, \chi_{-A'}}_l)| &\geq \\ &\geq \min(p, (k+l)|A'| - (k+l) + 2) - 2((\varepsilon_2 p)^{k+l-1} p)^{1/2}. \end{aligned}$$

If $|A'| \geq p/(k+l+1)$, we obtain

$$\begin{aligned} |\overline{S_{(\varepsilon_2 p)^{k+l-1}, k+l}(\underbrace{\chi_{A'}, \dots, \chi_{A'}}_k, \underbrace{\chi_{-A'}, \dots, \chi_{-A'}}_l)}| &= \\ = p - |S_{(\varepsilon_2 p)^{k+l-1}, k+l}(\underbrace{\chi_{A'}, \dots, \chi_{A'}}_k, \underbrace{\chi_{-A'}, \dots, \chi_{-A'}}_l)| &\leq \\ \leq p/(k+l+1) + 2\varepsilon_2^{(k+l-1)/2} p^{(k+l)/2} + (k+l-2). \end{aligned}$$

It is obvious that for any subset $B \subseteq \mathbf{Z}_p$ the set $kB - lB$ uniquely determines the set $\overline{kB - lB}$. From above it follows that the number of choices $kA - lA$ for given A' of cardinality exceeding $p/(k+l+1)$, is at most

$$2^{p/(k+l+1)+(k+l-2)+(2\varepsilon_2^{(k+l-1)/2} p^{(k+l-2)/2} + \varepsilon_3)p}. \quad (36)$$

If $|A'| < p/(k+l+1)$, then by (i) of Lemma 9 we have $|A \setminus A'| \leq \varepsilon_1 p$. This implies that $|A| \leq |A'| + \varepsilon_1 p$. Since every set $A \subseteq \mathbf{Z}_p$ generates exactly one set of form $kA - lA$, we obtain that the number of choices $kA - lA$ for given A' of cardinality not exceeding $p/(k+l+1)$, is at most

$$2^{p/(k+l+1)+\varepsilon_1 p}. \quad (37)$$

From (36), (37), Lemma 8 by applying Lemma 9 with parameters $\varepsilon_1 = \varepsilon_3 = \varepsilon$, $L = 1 + \lfloor 1/\varepsilon \rfloor$ and $\varepsilon_2 = \varepsilon^{2/(k+l-1)} p^{(2-k-l)/(k+l-1)}$, we obtain

$$|\mathbf{SS}_{k,l,s}''(\mathbf{Z}_p)| \leq 2^{(p/(k+l+1))+(k+l-2)+o(p)}. \quad (38)$$

From (33), (35) and (38) it follows that

$$|\mathbf{SS}_{k,l}(\mathbf{Z}_p)| \leq 2^{p/(k+l+1)} + 2^{(p/(k+l+1))+(k+l-2)+o(p)} = 2^{(p/(k+l+1))+(k+l-2)+o(p)}.$$

3.2 The lower bound

Set $\mathbf{SS}_{k,l}(\mathbf{Z}_p, \mathbb{P}) = \{A : \mathbb{P} \subseteq A, A \in \mathbf{SS}_{k,l}(\mathbf{Z}_p)\}$ and $L = \lfloor p/(2(k+l)-1) \rfloor - 1$.

Lemma 10 *Let k, l be nonnegative integers with $k+l \geq 2$, and let $\mathbb{P} \subseteq \mathbf{Z}_p$ be arbitrary arithmetic progression of length $(k+l)(L-1)+1$. Then there exists a positive constant $C_{k,l}$ such that*

$$|\mathbf{SS}_{k,l}(\mathbf{Z}_p, \mathbb{P})| \geq C_{k,l} 2^{p/(2(k+l)-1)}.$$

Proof. Without loss of generality we assume $\mathbb{P} = \{k-lL, \dots, kL-l\}$. All of our sets will be of the form

$$A = A(B) = k(B \cup \{-(2L+1), 2L+1\}) - l(B \cup \{-(2L+1), 2L+1\}),$$

where $B \subseteq \{-L, -L+1, \dots, L\}$ and $-B = B$. It is easy to see that different sets $B \subseteq \{-L, -L+1, \dots, L\}$ generate different sets $A(B)$.

Set $N_{k,l} = \lceil \log(8(k+l)^2) / \log(4/3) \rceil$ and

$$X = \{0, 1, \dots, N_{k,l}\} \cup \bigcup_{i=1}^{k+l-1} (\lfloor (i+1)L/(k+l) \rfloor - N_{k,l}, \dots, \lceil (i+1)L/(k+l) \rceil).$$

We define the set $B \subseteq \{-L, -L+1, \dots, L\}$ as follows:

$$B = B(C) = -C \cup C \cup X \cup -X,$$

where elements of the set C are picked from the set $\{1, \dots, L\} \setminus X$ randomly, independently, with probability $1/2$. Set

$$Y = \{0\} \cup \{k+l, \dots, (k+l)N_{k,l}\} \cup \bigcup_{i=1}^{k+l-1} \{(i+1)L - (k+l)N_{k,l}, \dots, (i+1)L\}.$$

It is obvious that $-Y \cup Y \subseteq kB - lB$. If $x \notin kB - lB$, then in the representation x in the form $x = x_1 + \dots + x_k - x_{k+1} - \dots - x_{k+l}$, there exists at least one x_i ($i \in \{1, \dots, k+l\}$) such that $x_i \notin B$. Set

$$\mathcal{Q}(x) = \{(x_1, \dots, x_{k+l}) : x = \sum_{i=1}^k x_i - \sum_{j=k+1}^{k+l} x_j, x_1, \dots, x_{k+l} \in \{-L, \dots, L\}\},$$

and suppose that $|\mathcal{Q}(x)| = q$.

We say that the vectors (x_1, \dots, x_{k+l}) and (y_1, \dots, y_{k+l}) do not intersect, if $\{x_1, \dots, x_{k+l}\} \cap \{y_1, \dots, y_{k+l}\} = \emptyset$.

Set $\mathcal{R}_0 = \{(k+l)N_{k,l} + 1, \dots, L\}$. We show that for every $x \in -\mathcal{R}_0 \cup \mathcal{R}_0$ the following inequality

$$\Pr(x \notin kB - lB) \leq \left(\frac{3}{4}\right)^{\left\lfloor \frac{|x|}{k+l} \right\rfloor} \quad (39)$$

holds. We have

$$\begin{aligned} & \Pr(x \notin kB - lB) = \\ &= \Pr((x_1^1 + \dots + x_k^1 - x_{k+1}^1 - \dots - x_{k+l}^1 \notin kB - lB) \& \dots \\ & \dots \& (x_1^q + \dots + x_k^q - x_{k+1}^q - \dots - x_{k+l}^q \notin kB - lB)) \leq \\ & \leq \Pr((x_1^{11} + \dots + x_k^{11} - x_{k+1}^{11} - \dots - x_{k+l}^{11} \notin kB - lB) \& \dots \\ & \dots \& (x_1^{1n} + \dots + x_k^{1n} - x_{k+1}^{1n} - \dots - x_{k+l}^{1n} \notin kB - lB)) = \\ &= \Pr\left((x_1^{11} \notin B \vee \dots \vee x_{k+1}^{11} \notin B) \& \dots \& (x_1^{1n} \notin B \vee \dots \vee x_{k+1}^{1n} \notin B)\right) = \\ &= \Pr\left((x_1^{11} \notin B) \vee \dots \vee (x_{k+1}^{11} \notin B)\right) \cdot \dots \cdot \Pr\left((x_1^{1n} \notin B) \vee \dots \vee (x_{k+1}^{1n} \notin B)\right) = \\ &= \Pr\left(\overline{(x_1^{11} \in B) \& \dots \& (x_{k+1}^{11} \in B)}\right) \times \dots \\ & \dots \times \Pr\left(\overline{(x_1^{1n} \in B) \& \dots \& (x_{k+1}^{1n} \in B)}\right) = \\ &= \left(1 - \Pr\left((x_1^{11} \in B) \& \dots \& (x_{k+1}^{11} \in B)\right)\right) \times \dots \\ & \dots \times \left(1 - \Pr\left((x_1^{1n} \in B) \& \dots \& (x_{k+1}^{1n} \in B)\right)\right), \end{aligned} \quad (40)$$

where the vectors $(x_1^i, \dots, x_{k+l}^i) \in \mathcal{Q}(x)$, $i = 1, \dots, q$, and the vectors $(x_1^{1j}, \dots, x_{k+l}^{1j})$, $j = 1, \dots, n \leq q$, are pairwise disjoint.

Note that the vectors $(x - i(k+l-1), \underbrace{i, \dots, i}_{k-1}, \underbrace{-i, \dots, -i}_l)$ are pairwise disjoint for every $x \in -\mathcal{R}_0$, where $-\lfloor |x|/(k+l) \rfloor \leq i \leq -1$, and $x \in \mathcal{R}_0$, where $1 \leq i \leq \lfloor |x|/(k+l) \rfloor$. From this and (40) we obtain the inequality (39).

Set $\mathcal{L}_j = \{jL+1, \dots, (j+1)L - (k+l)N_{k,l} - 1\}$, $j = 1, \dots, k+l-1$. Similarly to the inequality (39) we have

$$\Pr(x \notin kB - lB) \leq \left(\frac{3}{4}\right)^{\lfloor \frac{(j+1)L - |x|}{k+l} \rfloor}, \quad (41)$$

where $x \in -\mathcal{L}_j \cup \mathcal{L}_j$, $j = 1, \dots, k+l-1$.

From (39) and (41) it is easy to see that

$$\Pr(\mathbb{P} \not\subseteq kB - lB) \leq (k+l) \sum_{x \geq (k+l)N_{k,l}+1} \left(\frac{3}{4}\right)^{\lfloor \frac{x}{k+l} \rfloor}. \quad (42)$$

Note that if $N_{k,l} \geq \log(8(k+l)^2)/\log(4/3)$, the right-hand side of (42) does not exceed $1/2$. This leads that there exists at least $2^{L-(k+l)N_{k,l}-1}$ subsets $B \subseteq \{-L, -L+1, \dots, L\}$ such that $\mathbb{P} \subseteq kB - lB$. \square

Let k, l be nonnegative integers with $k+l \geq 2$, and let $\mathbb{P} \subseteq \mathbf{Z}_p$ be arbitrary arithmetic progression of length $(k+l)(L-1)+1$. By Lemma 10 we have

$$|\mathbf{SS}_{k,l}(\mathbf{Z}_p)| \geq |\mathbf{SS}_{k,l}(\mathbf{Z}_p, \mathbb{P})| \geq C_{k,l} 2^{p/(2(k+l)-1)}.$$

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